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The Weierstrass–Stone theorem for convex-cone-valued functions

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Abstract

Theorems of the Weierstrass–Stone-type are presented for the convex cone of all continuous functions defined on a compact Hausdorff space S with values in a convex cone $\mathscr{C}$ equipped with a suitable metric d. These results are applied to establish a Bohman–Korovkin-type theorem for monotone approximation.

Keywords: Convex cone; Hausdorff metric

1. Metric convex cones

Definition 1.1. An (abstract) *convex cone* is a nonempty set $\mathscr{C}$ such that to every pair of elements K and L of $\mathscr{C}$ there corresponds an element $K+L$, called the *sum* of K and L, in such a way that addition is commutative and associative, and there exists in $\mathscr{C}$ a unique element 0, called the *vertex* of $\mathscr{C}$, such that $K+0=K$, for every $K\in\mathscr{C}$. Moreover, to every pair λ and K, where $\lambda\geqslant 0$ is a nonnegative real number and $K\in\mathscr{C}$, there corresponds an element λK, called the *product* of λ and K, in such a way that multiplication is associative: $\lambda(\mu K)=(\lambda\mu)K$; $1\cdot K=K$ and $0\cdot K=0$ for every $K\in\mathscr{C}$; and the distributive laws are verified: $\lambda(K+L)=\lambda K+\lambda L$, $(\lambda+\mu)K=\lambda K+\mu K$, for every $K, L\in\mathscr{C}$ and $\lambda\geqslant 0$, $\mu\geqslant 0$.

A nonempty subset $\mathscr{K}$ of an (abstract) convex cone $\mathscr{C}$ is called a *convex subcone* if $K, L\in\mathscr{K}$ and $\lambda\geqslant 0$ imply $K+L\in\mathscr{K}$ and $\lambda K\in\mathscr{K}$.

Definition 1.2. An *ordered convex cone* is a pair $(\mathscr{C}, \leqslant)$, where $\mathscr{C}$ is an (abstract) convex cone and $\leqslant$ is an ordering of its elements, i.e., $\leqslant$ is a reflexive, transitive and antisymmetric relation on $\mathscr{C}$, in such a way that

$$K\leqslant L \text{ implies } K+M\leqslant L+M,$$
$$K\leqslant L,\ \lambda\geqslant 0 \text{ implies } \lambda K\leqslant\lambda L,$$
$$K\leqslant K+L,\quad \text{for every } L\geqslant 0.$$

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Example 1.3. Let E be a vector space over the reals. Let $\mathscr{C} = \text{Conv}(E)$ be the set of all convex nonempty subsets of E. If K, $L \in \text{Conv}(E)$ and $\lambda \geqslant 0$, define

$$K + L = \{u + v;\ u \in K,\ v \in L\},$$
$$\lambda K = \{\lambda u;\ u \in K\},$$
$$0 = \{\theta\}, \quad \text{where } \theta \text{ is the origin of } E,$$
$$K \leqslant L \quad \text{if, and only if,} \quad K \subset L.$$

With this definition, $(\text{Conv}(E), \leqslant)$ is an ordered convex cone.

Example 1.4. Let S be a nonempty set and let $(\mathscr{C}, \leqslant)$ be an ordered convex cone. The set $\mathscr{F}(S; \mathscr{C})$ of all mappings $F: S \to \mathscr{C}$, with pointwise operations and ordering, is an ordered convex cone.

Definition 1.5. Let $\mathscr{C}$ be an (abstract) convex cone and let d be a metric on $\mathscr{C}$ (respectively a semi-metric on $\mathscr{C}$). We say that the pair $(\mathscr{C}, d)$ is a *metric convex cone* (respectively a *semi-metric convex cone*) if the following properties are valid:

$$\text{(a)} \qquad d\left(\sum_{i=1}^{m} K_i, \sum_{i=1}^{m} L_i\right) \leqslant \sum_{i=1}^{m} d(K_i, L_i),$$

$$\text{(b)} \qquad d(\lambda K, \lambda L) = \lambda d(K, L),$$

for every K_i, L_i, $i = 1, \ldots, m$, K, L in $\mathscr{C}$ and every $\lambda \geqslant 0$.

Definition 1.6. Let $(\mathscr{C}, \leqslant)$ be an ordered convex cone and let d_H be a semi-metric on $\mathscr{C}$. We say that d_H is a *Hausdorff semi-metric* on $\mathscr{C}$ if the following condition holds: there exists an element $B \geqslant 0$ on $\mathscr{C}$ such that

(a) for every pair K, $L \in \mathscr{C}$ and $\lambda > 0$, the following is true: $d_H(K, L) \leqslant \lambda$ if, and only if, $K \leqslant L + \lambda B$ and $L \leqslant K + \lambda B$;

(b) $\lambda B \leqslant \mu B$ implies $\lambda \leqslant \mu$.

If d_H is a Hausdorff semi-metric on $\mathscr{C}$, we say that $(\mathscr{C}, d_H)$, or $\mathscr{C}$, is a *Hausdorff convex cone*.

Example 1.7. If $\mathscr{C} = \mathbb{R}_+$ with the usual operations and ordering, then the usual distance $d_H(x, y) = |x - y|$ is a Hausdorff metric on $\mathbb{R}_+$, with $B = 1$. Notice that we can also take $\mathscr{C} = \mathbb{R}$, and the usual distance is still a Hausdorff metric on $\mathbb{R}$.

Example 1.8. Let E be a normed space over the reals. Let $\mathscr{C}(E)$ be the convex subcone of $\text{Conv}(E)$, consisting of those elements of $\text{Conv}(E)$ that are *bounded* sets, and let B be the closed unit ball of E. Define on $\mathscr{C}(E)$ the usual Hausdorff semi-metric d_H by setting

$$d_H(K, L) = \inf\{\lambda > 0;\ K \subset L + \lambda B,\ L \subset K + \lambda B\},$$

for every pair K, $L \in \mathscr{C}(E)$. Then $(\mathscr{C}(E), d_H)$ is a Hausdorff convex cone. If $\mathscr{K} = \mathscr{K}(E)$, the set of all *compact* and convex nonempty subsets of E, then $\mathscr{K}$ is a convex subcone of $\mathscr{C}(E)$.

Proposition 1.9. *Let* $(\mathscr{C}, d)$ *be either a metric or a semi-metric convex cone. Then,*

$$(1) \qquad d(\lambda K, \mu K) \leqslant |\lambda - \mu| d(K, 0),$$

$$(2) \qquad d(\lambda K, \mu L) \leqslant |\lambda - \mu| d(K, 0) + \mu d(K, L),$$

for every K *and* L *in* $\mathscr{C}$ *and every* $\lambda \geqslant 0$ *and* $\mu \geqslant 0$.

Proof. To prove (1) we may assume $\mu < \lambda$. Write $\lambda K = \mu K + (\lambda - \mu)K$ and $\mu K = \mu K + (\lambda - \mu)0$. By (a) and (b) of Definition 1.5 we get

$$d(\lambda K, \mu K) \leqslant d(\mu K, \mu K) + d((\lambda - \mu)K, (\lambda - \mu)0) = |\lambda - \mu| d(K, 0).$$

Property (2) follows from (b) and (1) together with the triangle inequality. □

Proposition 1.10. *Let* $(\mathscr{C}, d_{\mathrm{H}})$ *be a Hausdorff convex cone. Then,*

$$(1) \qquad d_{\mathrm{H}}\left(\sum_{i=1}^{m} K_i, \sum_{i=1}^{m} L_i\right) \leqslant \sum_{i=1}^{m} d_{\mathrm{H}}(K_i, L_i),$$

$$(2) \qquad d_{\mathrm{H}}(\lambda K, \lambda L) = \lambda d_{\mathrm{H}}(K, L),$$

for every K_i, L_i, $i = 1, \ldots, m$, K *and* L *in* $\mathscr{C}$ *and* $\lambda \geqslant 0$.

Proof. To prove (1), let $d_i = d_{\mathrm{H}}(K_i, L_i)$ for $i = 1, \ldots, m$. If $\epsilon > 0$ is given, then

$$K_i \leqslant L_i + \left(d_i + \frac{\epsilon}{m}\right)B, \qquad L_i \leqslant K_i + \left(d_i + \frac{\epsilon}{m}\right)B,$$

for all $i = 1, \ldots, m$. Hence,

$$\sum_{i=1}^{m} K_i \leqslant \sum_{i=1}^{m} L_i + \left(\sum_{i=1}^{m} d_i + \epsilon\right)B, \qquad \sum_{i=1}^{m} L_i \leqslant \sum_{i=1}^{m} K_i + \left(\sum_{i=1}^{m} d_i + \epsilon\right)B.$$

From this it follows that

$$d_{\mathrm{H}}\left(\sum_{i=1}^{m} K_i, \sum_{i=1}^{m} L_i\right) \leqslant \sum_{i=1}^{m} d_i + \epsilon.$$

Since $\epsilon > 0$ was arbitrary, property (1) follows.

The proof of (2) is similar, and therefore we omit it. □

Let S be a compact Hausdorff space. Let $(\mathscr{C}, d)$ be a semi-metric convex cone. We denote by $C(S; \mathscr{C})$ the convex subcone of $\mathscr{F}(S; \mathscr{C})$ consisting of all continuous mappings $F: S \to \mathscr{C}$. In $C(S; \mathscr{C})$ we consider the topology of uniform convergence over S, determined by the semi-metric defined by

$$d(F, G) = \sup\{d(F(s), G(s)); s \in S\},$$

for every pair F, G of elements of $C(S; \mathscr{C})$. Hence $F_n \to F$ in $C(S; \mathscr{C})$ if, and only if, $d(F_n, F) \to 0$.

When $(\mathscr{C}, d)$ is $\mathbb{R}$ equipped with the usual distance $d(x, y) = |x - y|$, then $C(S; \mathscr{C})$ is the classical Banach space $C(S)$ of all continuous real-valued functions $f: S \to \mathbb{R}$, equipped with the sup-norm

$$\| f \| = \sup\{ | f(s) |;\ s \in S\}.$$

When $(\mathscr{C}, d)$ is $\mathbb{R}_+$ equipped with the usual distance, then $C(S; \mathbb{R}_+) = C_+(S)$, where $C_+(S) = \{f \in C(S);\ f \geqslant 0\}$.

Proposition 1.11. *If $\phi \in C_+(S)$ and $F \in C(S; \mathscr{C})$, then the function $s \mapsto \phi(s)F(s)$, $s \in S$, belongs to $C(S; \mathscr{C})$. In particular, for every $\phi \in C_+(S)$ and every $K \in \mathscr{C}$, the function $s \mapsto \phi(s)K$, $s \in S$, belongs to $C(S; \mathscr{C})$.*

Proof. Put $\| F \| = \sup\{d(F(s), 0);\ s \in S\}$. Since $F \in C(S; \mathscr{C})$, it follows that $\| F \| < \infty$. By Proposition 1.9, we have

$$d(\phi(s)F(s), \phi(t)F(t)) \leqslant |\phi(s) - \phi(t)| \, \| F \| + \| \phi \| \, d(F(s), F(t)),$$

for every pair s and t of elements of S. From this the continuity of the function ϕF follows now from the continuity of ϕ and F. □

2. A property introduced by von Neumann

Definition 2.1. A subset $M \subset C(S; [0, 1])$ is said to have property V if
(1) $\phi \in M$ implies $1 - \phi \in M$;
(2) $\phi, \psi \in M$ implies $\phi\psi \in M$.

Property V was introduced by von Neumann, who stated in [9], without proof, that a subset $M \subset C([0, 1]^n; [0, 1])$ containing a constant $0 < c < 1$, containing the n projections, and having property V, is dense. In his paper [3], Jewett provided a proof for von Neumann's result and described the closure of a point-separating $M \subset C(S; [0, 1])$ having property V. In [5], we removed the hypothesis of M being point-separating, and described the closure of any subset that has property V.

We recall that given any set M of continuous real-valued functions on S, the equivalence relation $x \equiv y \pmod M$ is defined as follows: for any pair of points $x, y \in S$ we declare $x \equiv y \pmod M$ if, and only if, $\phi(x) = \phi(y)$ for all $\phi \in M$. For each $x \in S$, it follows that the equivalence class $[x] \pmod M$ is a closed subset of S:

$$[x] = \{t \in S;\ \phi(t) = \phi(x), \text{ for all } \phi \in M\} = \bigcap_{\phi \in M} \phi^{-1}(\phi(x)).$$

Hence $[x]$ is a compact subset of S, for each $s \in S$.

Lemma 2.2. *Let $M \subset C(S; [0, 1])$ be a nonempty set having property V. Let $x \in S$ and let $N(x)$ be an open neighborhood of the equivalence class $[x] \pmod M$. There exists an open neighborhood $U(x)$ of $[x]$, contained in $N(x)$, such that, for each $0 < \delta < \frac{1}{2}$, there is $\phi \in M$ such that*

$$(1) \qquad \phi(t) > 1 - \delta, \quad \text{for all } t \in U(x),$$

$$(2) \qquad 0 \leqslant \phi(t) < \delta, \quad \text{if } t \notin N(x).$$

Proof. Apply [4, Lemma 1], with $\frac{1}{2}\delta$, and then uniformly approximate within $\frac{1}{2}\delta$ the function obtained in $\bar{M}$, by an element $\phi \in M$. □

3. Weierstrass–Stone theorems

Definition 3.1. Let W be a nonempty subset of $C(S; \mathscr{C})$. A function $\phi \in C(S; [0, 1])$ is called a *multiplier* of W if $F, G \in W$ implies $\phi F + (1 - \phi)G \in W$.

It is clear that if ϕ is a multiplier of W, then $1 - \phi$ is a multiplier of W. Suppose now that ϕ and ψ are multipliers of W. The identity

$$1 - \phi\psi = (1 - \phi) + \phi(1 - \psi)$$

implies that, for every pair $F, G \in W$,

$$(\phi\psi)F + (1 - \phi\psi)G = \phi[\psi F + (1 - \psi)G] + (1 - \phi)G,$$

and therefore $\phi\psi$ is a multiplier of W. Hence, the set of all multipliers of W has property V.

Theorem 3.2. Let $(\mathscr{C}, d)$ *be a metric or a semi-metric convex cone and let* W *be a nonempty subset of* $C(S; \mathscr{C})$. *Let* $M \subset C(S; [0, 1])$ *be the set of all multipliers of* W. *For each* $F \in C(S; \mathscr{C})$ *and each* $\epsilon > 0$, *the following are equivalent*:

(1) *there is some* $G \in W$ *such that* $d(F, G) < \epsilon$;

(2) *for each* $x \in S$, *there is some* $G_x \in W$ *such that* $d(F(t), G_x(t)) < \epsilon$, *for all* $t \in [x]$.

Proof. Clearly, (1) ⇒ (2). Conversely, assume that (2) is true. For each $x \in S$, there is some $G_x \in W$ such that $d(F(t), G_x(t)) < \epsilon$ for all t in the equivalence class $[x]$. Now $[x]$ is a compact subset of S, and therefore we can select a real number $\epsilon(x) > 0$ such that $d(F(t), G_x(t)) < \epsilon(x) < \epsilon$ for all $t \in [x]$. Then,

$$N(x) = \{t \in S;\ d(F(t), G_x(t)) < \epsilon(x)\}$$

is an open neighborhood of $[x]$. By Lemma 2.2, choose an open neighborhood $U(x)$ with the property stated in Lemma 2.2. Select a point $x_1 \in S$ arbitrarily, and let K be the complement of $N(x_1)$ in S. By compactness of K, there is a finite set $\{x_2, \ldots, x_m\} \subset K$ such that $K \subset U(x_2) \cup \cdots \cup U(x_m)$. Choose $0 < \delta < \frac{1}{2}$ so small that $\delta k m < \epsilon - \epsilon'$, where

$$\epsilon' = \max\{\epsilon(x_i);\ 1 \leqslant i \leqslant m\}, \qquad k = \max\{d(F, G_{x_i});\ 1 \leqslant i \leqslant m\}.$$

By Lemma 2.2, there are $\phi_2, \ldots, \phi_m \in M$ such that for each $i = 2, \ldots, m$,

$$(1) \qquad \phi_i(x) > 1 - \delta, \quad \text{for all } x \in U(x_i),$$
$$(2) \qquad 0 \leqslant \phi_i(t) < \delta, \quad \text{if } t \notin N(x_i).$$

Define

$$\psi_2 = \phi_2,$$
$$\psi_3 = (1 - \phi_2)\phi_3,$$
$$\vdots$$
$$\psi_m = (1 - \phi_2)(1 - \phi_3) \cdots (1 - \phi_{m-1})\phi_m.$$

Clearly, $\psi_i \in M$, for all $i = 2, \ldots, m$. Now,

$$\psi_2 + \cdots + \psi_j = 1 - (1 - \phi_2) \cdot \ldots \cdot (1 - \phi_j), \quad j = 2, \ldots, m,$$

can be easily verified by induction. Define

$$\psi_1 = (1 - \phi_2) \cdot \ldots \cdot (1 - \phi_m).$$

Then $\psi_1 \in M$, and $\psi_1 + \psi_2 + \cdots + \psi_m = 1$. Notice that

$$(3) \qquad \psi_i(t) < \delta, \quad \text{for all } t \notin N(x_i), \ i = 1, \ldots, m.$$

Indeed, if $i \geqslant 2$, $\psi_i(t) \leqslant \phi_i(t)$ and (3) follows from (2). If $i = 1$, and $t \notin N(x_1)$, then $t \in K$. Hence $t \in U(x_j)$, for some $j = 2, \ldots, m$. By (1), $1 - \phi_j(t) < \delta$ and so

$$\psi_1(t) = (1 - \phi_j(t)) \prod_{i \neq j} (1 - \phi_i(t)) < \delta.$$

Let $G = \psi_1 G_1 + \psi_2 G_2 + \cdots + \psi_m G_m$, where $G_i = G_{x_i}$, $i = 1, \ldots, m$. Notice that

$$G = \phi_2 G_2 + (1 - \phi_2)[\phi_3 G_3 + (1 - \phi_3)[\phi_4 G_4 + \cdots + (1 - \phi_{m-1})[\phi_m G_m + (1 - \phi_m) G_1] \cdots]].$$

Hence $G \in W$. For each $x \in S$, we have

$$(4) \qquad d(F(x), G(x)) = d\left(\sum_{i=1}^{m} \psi_i(x) F(x), \sum_{i=1}^{m} \psi_i(x) G_i(x) \right) \leqslant \sum_{i=1}^{m} \psi_i(x) d(F(x), G_i(x)).$$

Let $I = \{1 \leqslant i \leqslant m;\ x \in N(x_i)\}$ and $J = \{1 \leqslant i \leqslant m;\ x \notin N(x_i)\}$. Then, for all $i \in I$ we have

$$(5) \qquad \psi_i(x) d(F(x), G_i(x)) \leqslant \psi_i(x) \epsilon'.$$

And, for all $i \in J$, we have by (3),

$$(6) \qquad \psi_i(x) d(F(x), G_i(x)) \leqslant \delta k.$$

Hence (5) and (6) imply

$$(7) \qquad \sum_{i=1}^{m} \psi_i(x) d(F(x), G_i(x)) \leqslant \sum_{i \in I} \psi_i(x) \epsilon' + \sum_{i \in J} \delta k \leqslant \epsilon' + \delta k m < \epsilon.$$

From (4) and (7) we get the estimate $d(F(x), G(x)) < \epsilon$. □

Before stating our next Weierstrass–Stone theorem, we introduce some notation. Let $F \in C(S; \mathscr{E})$ and $x \in S$. Let $[x]$ be the equivalence class of the point x (mod M), where $M \subset C(S; [0, 1])$. Then $F[x]$ denotes the restriction of F to the closed subset $[x] \subset S$. Clearly, $F[x]$ belongs to $C([x]; \mathscr{E})$, for each $x \in S$. If $W \subset C(S; \mathscr{E})$ is any subset, then

$$W[x] = \{G[x];\ G \in W\}.$$

Clearly, $W[x] \subset C([x]; \mathscr{C})$ and

$$\text{dist}(F[x]; W[x]) = \inf\{d(F[x], G[x]); G \in W\}$$

and

$$d(F[x], G[x]) = \sup\{d(F(t), G(t)); t \in [x]\}.$$

Theorem 3.3. *Let W be a nonempty subset of $C(S; \mathscr{C})$, and let M be the set of all multipliers of W. For each $F \in C(S; \mathscr{C})$ there is some $x \in S$ such that*

$$\text{dist}(F; W) = \text{dist}(F[x]; W[x]).$$

Proof. If $\text{dist}(F; W) = 0$, then $\text{dist}(F[x]; W[x]) = 0$ for every $x \in S$. Suppose now $\text{dist}(F; W) = d > 0$. Assume, by contradiction, that $\text{dist}(F[x], W[x]) < d$, for each $x \in S$. Then, there exists $G_x \in W$ such that $d(F(t), G_x(t)) < d$, for all $t \in [x]$. Consequently, F and d satisfy condition (2) of Theorem 3.2. By the equivalence between conditions (1) and (2) of Theorem 3.2, there is some $G \in W$ such that $d(F, G) < d = \text{dist}(F; W)$, a contradiction. □

Corollary 3.4. *Let $W \subset C(S; \mathscr{C})$ be a nonempty subset such that the set of all multipliers of W separates the points of S. For each $F \in C(S; \mathscr{C})$, there is some $x \in S$ such that*

$$\text{dist}(F; W) = \text{dist}(F(x); W(x)).$$

Proof. Immediate from Theorem 3.3, since each equivalence class $[x]$ (mod M) reduces to the singleton $\{x\}$. □

Corollary 3.5. *Let $W \subset C(S; \mathscr{C})$ be a nonempty subset and let M be the set of all multipliers of W. For each $F \in C(S; \mathscr{C})$, the following are equivalent.*

(1) *F belongs to the uniform closure of W in $C(S; \mathscr{C})$.*

(2) *For each $x \in S$, the restriction of F to $[x]$ (mod M) belongs to the uniform closure of $W[x]$ in $C([x]; \mathscr{C})$.*

Proof. The implication (1) ⇒ (2) is clear. Conversely, if F satisfies (2), then, for each $x \in S$, $\text{dist}(F[x]; W[x]) = 0$. It remains to apply Theorem 3.3 to conclude that $\text{dist}(F; W) = 0$. Therefore (1) is verified, and so (2) ⇒ (1). □

Collary 3.6. *Let $W \subset C(S; \mathscr{C})$ be a nonempty subset. Assume that*

(1) *for each pair of distinct points x and y of S, there is some multiplier ϕ of W such that $\phi(x) \neq \phi(y)$;*

(2) *for each point $x \in S$ and each $K \in \mathscr{C}$, there is some $G \in W$ such that $G(x) = K$.*

Then W is dense in $C(S; \mathscr{C})$.

Proof. Let $F \in C(S; \mathscr{C})$ be given. By Corollary 3.4, there is some $x \in S$ such that

$$\text{dist}(F; W) = \text{dist}(F(x); W(x)).$$

Now, by (2), there is some $G \in W$ such that $G(x) = F(x)$. Hence, $\text{dist}(F(x); W(x)) = 0$. Consequently, $\text{dist}(F; W) = 0$, for all functions $F \in C(S; \mathscr{C})$, and so W is dense in $C(S; \mathscr{C})$. □

4. Applications

Example 4.1. Let S be a compact Hausdorff space, and let $(\mathscr{C}, d)$ be a semi-metric convex cone. Let W be the convex subcone of $C(S; \mathscr{C})$ generated by the functions of the form

$$t \mapsto f(t)K, \quad t \in S,$$

where $f \in C(S; \mathbb{R})$ is nonnegative and $K \in \mathscr{C}$. Now W contains the constant functions and each $\phi \in C(S; \mathbb{R})$, $0 \leqslant \phi \leqslant 1$, is a multiplier of W. Hence, by Corollary 3.6, W is dense in $C(S; \mathscr{C})$. Notice that the elements of W are of the form

$$t \mapsto \sum_{i=1}^{n} f_i(t)K_i,$$

where $K_1, \ldots, K_n \in \mathscr{C}$ and $f_1, \ldots, f_n \in C(S; \mathbb{R}_+)$ and $n = 1, 2, 3, \ldots$.

Example 4.2. Let S and $\mathscr{C}$ be as in Example 4.1. Let $\phi \in C(S; \mathbb{R})$ be such that $0 \leqslant \phi \leqslant 1$. Let W be the convex subcone of $C(S; \mathscr{C})$ generated by the functions of the form

$$t \mapsto (\phi(t))^i (1 - \phi(t))^j K, \quad t \in S,$$

where $i, j = 0, 1, 2, 3, \ldots,$ and $K \in \mathscr{C}$. Clearly, W contains the constant functions and ϕ is a multiplier of W. Hence, if ϕ is one-to-one, then W is dense in $C(S; \mathscr{C})$. Notice that the elements of W are of the form

$$t \mapsto \sum_{i+j \leqslant n} (\phi(t))^i (1 - \phi(t))^j K_{ij},$$

where $K_{ij} \in \mathscr{C}$, $i, j = 0, 1, 2, \ldots,$ $n = 0, 1, 2, \ldots$.

Example 4.3. Let S and C be as in Example 4.1, and let $\mathscr{K} \subset \mathscr{C}$ be a convex subcone. Let $W = \{F \in C(S; \mathscr{C});\ F(t) \in \mathscr{K}$ for all $t \in T\}$, where T is some closed and nonempty subset of S. Clearly, W is a convex subcone of $C(S; \mathscr{C})$ and any $\phi \in C(S; \mathbb{R})$, $0 \leqslant \phi \leqslant 1$, is a multiplier of W. On the other hand, for each $t \in T$, $W(t) = \mathscr{K}$; and for each $t \in S$, $t \notin T$, $W(t) = \mathscr{C}$. Hence Theorem 3.2 implies that, for any $F \in C(S; \mathscr{C})$,

$$\operatorname{dist}(F; W) = \sup_{t \in T} \operatorname{dist}(F(t); \mathscr{K}),$$

where, for each $t \in T$,

$$\operatorname{dist}(F(t); \mathscr{K}) = \inf\{d(F(t), K);\ K \in \mathscr{K}\}.$$

Example 4.4. Let $S = [0, 1]$ and let $(\mathscr{C}, d)$ be a semi-metric convex cone. Let W be the convex subcone of $C([0, 1]; \mathscr{C})$ generated by the functions of the form

$$t \mapsto t^i (1 - t)^j K, \quad 0 \leqslant t \leqslant 1,$$

where $i, j = 0, 1, 2, \ldots,$ and $K \in \mathscr{C}$. Clearly, W contains all the constant functions and, moreover, the function $\phi(t) = t$ is a multiplier of W. Hence, by Corollary 3.6, W is dense in $C([0, 1]; \mathscr{C})$. Notice that the elements of W are of the form

$$t \mapsto \sum_{i,j=0}^{n} t^i (1 - t)^j K_{i,j},$$

where $K_{i,j} \in \mathscr{C}$, $i, j = 0, 1, \ldots, n$, and $n = 0, 1, 2, 3\ldots$. A similar result holds for any interval $S = [a, b]$, with $0 \leqslant a < b$. In [6] we showed that when $\mathscr{C}$ is a Hausdorff convex cone, the Bernstein operators B_n can be used to produce a sequence $\{B_n F\}_{n \geqslant 1}$ of elements in W such that $B_n F \to F$, for each $F \in C([0, 1]; \mathscr{C})$, where for $0 \leqslant t \leqslant 1$,

$$(B_n F, t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} F\left(\frac{k}{n}\right).$$

Example 4.5 Let S be a compact subset of $\mathbb{R}^n$, contained in $[0, 1]^n$. Let now W be the convex subcone of $C(S; \mathscr{C})$ generated by the functions of the form

$$(t_1, \ldots, t_n) \mapsto t_1^{i_1}(1-t_1)^{j_1} \cdot \ldots \cdot t_n^{i_n}(1-t_n)^{j_n} K,$$

where i_k, $j_k = 0, 1, 2, \ldots$, $k = 1, \ldots, n$ and $K \in \mathscr{C}$. Since all the projections $\pi_j : S \to \mathbb{R}_+$ are multipliers of W, the set of all multipliers of W separates the points of S, and Corollary 3.6 can again be applied to conclude that the convex subcone W is dense in $C(S; \mathscr{C})$.

5. Ransford's proof

A different proof of Theorem 3.3 for *arbitrary* subset W of $C(S; \mathscr{C})$ is possible if we use Ransford's idea (see [1,8]). When T is an arbitrary nonempty closed subset of S, we denote by F_T the restriction of F to T, if $F \in C(S; \mathscr{C})$, and then $W_T = \{F_T;\ F \in W\}$, for any subset $W \subset C(S; \mathscr{C})$. Clearly, $F_T \in C(T; \mathscr{C})$ and $W_T \subset C(T; \mathscr{C})$. Obviously, if $T = [x]$, then $F_T = F[x]$, and $W_T = W[x]$.

Lemma 5.1. *Given any nonempty subset $W \subset C(S; \mathscr{C})$ and any $F \in C(S; \mathscr{C})$, there is a minimal closed and nonempty set $T \subset S$ such that*

$$\operatorname{dist}(F; W) = \operatorname{dist}(F_T; W_T).$$

Proof. Let $d = \operatorname{dist}(F; W)$. If $d = 0$, then any singleton $T = \{x\}$, for $x \in S$, also satisfies $t(F_T; W_T) = 0$. Hence, we may assume $d > 0$.

Let $\mathscr{F}$ be the family of all closed nonempty subsets $T \subset S$ such that

$$\operatorname{dist}(F_T; W_T) = d.$$

Since $S \in \mathscr{F}$, we see that $\mathscr{F}$ is nonempty. We order the family $\mathscr{F}$ by set inclusion. Let $\mathscr{G}$ be a totally ordered subfamily of $\mathscr{F}$, and let T be the intersection of all members of $\mathscr{G}$. By compactness of S, it follows that $T \neq \emptyset$. We claim that $T \in \mathscr{F}$. Indeed, assume that $T \notin \mathscr{F}$. Then $\operatorname{dist}(F_T; W_T) < d$. Choose r such that $\operatorname{dist}(F_T; W_T) < r < d$. Consequently, $d(F_T; G_T) < r$ for some $G \in W$. Hence $U = \{t \in S;\ d(F(t), G(t)) < r\}$ is an open subset of S, containing T. By compactness, there is a finite family $T_1, \ldots, T_n$ of elements of $\mathscr{G}$ such that $T_1 \cap \cdots \cap T_n \subset U$. Since the family $\mathscr{G}$ is totally ordered, there is some index $k \in \{1, \ldots, n\}$ such that $T_k = T_1 \cap \cdots \cap T_n$. Hence $T_k \subset U$, and therefore $d(F_{T_k}, G_{T_k}) \leqslant r < d$, which contradicts the fact that $T_k \in \mathscr{F}$. Hence $T \in \mathscr{F}$. By Zorn's Lemma, there is a minimal element of $\mathscr{F}$. $\square$

Second proof of Theorem 3.3. Let T be given by Lemma 5.1. The result will follow if we show that $T \subset [x]$, for some point $x \in S$. By contradiction, assume that this is false. Hence, there is some multiplier $\phi \in M$ which is not constant on T, and therefore $\phi(z) > \phi(y)$ for some pair z and y of elements of T. Choose $a < b$ such that $\phi(y) < a < b < \phi(z)$. Define

$$Y = T \cap \phi^{-1}([0, b]), \qquad Z = T \cap \phi^{-1}([a, 1]).$$

Notice that $T = Y \cup Z$ and that both Y and Z are nonempty proper closed subsets of T. By minimality of T, there exist functions $v, w \in W$ such that $d(F_Y, v_Y) < d$ and $d(F_Z, w_Z) < d$. Choose $0 < \epsilon < 1$ so that $\epsilon < d - d(F_Y, v_Y)$ and $\epsilon < d - d(F_Z, w_Z)$. By compactness of S, there is some constant $0 < k < 1$ so that $d(v(t), 0) \leqslant k$ and $d(w(t), 0) \leqslant k$, for all $t \in S$. Choose a positive integer r so that $(\frac{2}{3})^r < \epsilon/(2k)$. Next choose a positive integer m such that

$$\frac{1}{2}\frac{1}{b^m} > 1 \quad \text{and} \quad \left(\frac{b}{a}\right)^m \frac{\epsilon}{2k}\frac{1}{r} > 1.$$

Then, choose a positive integer s such that

$$\frac{1}{2}\frac{1}{b^m} \leqslant s < \frac{1}{b^m}.$$

Let $n = rs$. Then,

$$n < \frac{r}{b^m} < \frac{1}{a^m}\frac{\epsilon}{2k} \quad \text{and} \quad sb^m \geqslant \tfrac{1}{2}.$$

Hence, for any $0 \leqslant x \leqslant a$, we have by Bernoulli's inequality,

$$(1) \qquad (1 - x^m)^n \geqslant (1 - a^m)^n \geqslant 1 - na^m > 1 - \frac{\epsilon}{2k}.$$

For $1 \geqslant x \geqslant b$, again by Bernoulli's inequality, we have

$$(2) \qquad (1 - x^m)^n \leqslant (1 - b^m)^n \leqslant \left[(1 + b^m)^{-s}\right]^r \leqslant \left[(1 + sb^m)^{-1}\right]^r \leqslant \tfrac{2}{3}r < \frac{\epsilon}{2k}.$$

Let $h(t) = p(\phi(t))$, for all $t \in S$, where $p(x) = (1 - x^m)^n$, for all $x \in [0, 1]$. Since M has property V, the function h is a multiplier of W and therefore $G \in W$, if we define

$$(3) \qquad G = hv + (1 - h)w.$$

Let $t \in Y \cap Z$. Then $d(F(t), v(t)) < d$ and $d(F(t), w(t)) < d$. Hence,

$$(4) \qquad d(F(t), G(t)) < d,$$

for all $t \in Y \cap Z$, because $F = hF + (1 - h)F$.

Let $t \in T$, $t \notin Z$. Then $t \in Y$ and $\phi(t) < a$. Consequently, (1) implies

$$h(t) = p(\phi(t)) > 1 - \frac{\epsilon}{2k}.$$

The identity $v = hv + (1 - h)v$ implies

$$d(G(t), v(t)) = d(h(t)v(t) + (1 - h(t))w(t), v(t))$$
$$\leqslant |1 - h(t)|\, d(w(t), v(t)) < \frac{\epsilon}{2k}2k = \epsilon,$$

and, since $t \in Y$,

$$(5) \qquad d(F(t), G(t)) \leqslant d(F(t), v(t)) + d(v(t), G(t)) \leqslant d(F_Y, v_Y) + \epsilon < d.$$

Finally, let $t \in T$, $t \notin Y$. Then $t \in Z$ and $\phi(t) > b$. Consequently, by (2),

$$h(t) = p(\phi(t)) \leqslant \frac{\epsilon}{2k}.$$

The identity $w = hw + (1-h)w$ implies

$$\begin{aligned} d(G(t), w(t)) &= d(h(t)v(t) + (1-h(t))w(t), w(t)) \\ &\leqslant h(t)d(v(t), w(t)) < \frac{\epsilon}{2k}2k = \epsilon, \end{aligned}$$

and, since $t \in Z$,

$$(6) \qquad d(F(t), G(t)) \leqslant d(F(t), w(t)) + d(w(t), G(t)) \leqslant d(F_Z, w_Z) + \epsilon < d.$$

Now $T = (Y \cap Z) \cup (T \setminus Z) \cup (T \setminus Y)$, and (4)–(6) imply $d(F(t), G(t)) < d$ for all $t \in S$, contradicting the fact that $T \in \mathscr{F}$. □

6. Korovkin systems

Definition 6.1. Let $\mathscr{K}$ be a convex subcone of $\mathscr{C}$, and let $\mathscr{A}$ be a class of $\mathbb{R}_+$-linear operators on $C(S; \mathscr{C})$. A subset $\mathscr{G} \subset C(S; \mathscr{K})$ is called a *Korovkin system* in $C(S; \mathscr{K})$ for $\mathscr{A}$ if for every uniformly equicontinuous sequence $\{T_n\}_{n \geqslant 1}$ of $\mathbb{R}_+$-linear operators belonging to $\mathscr{A}$ the following holds:

$(*)$ $T_n G \to G$, for all $G \in \mathscr{G}$, implies $T_n F \to F$, for all $F \in C(S; \mathscr{K})$.

When $(*)$ holds only for sequences consisting of operators that are monotonically regular over $\mathscr{K}$, then we say that $\mathscr{G}$ is a *regular Korovkin system*.

We recall that an $\mathbb{R}_+$-linear operator T on $C(S; \mathscr{C})$ is a function $T : C(S; \mathscr{C}) \to C(S; \mathscr{C})$ such that

(a) $T(F+G) = T(F) + T(G)$,

(b) $T(\lambda F) = \lambda T(F)$,

for all $F, G \in C(S; \mathscr{C})$ and $\lambda \geqslant 0$. Such an operator is called *monotonically regular* over $\mathscr{K}$ is there exists a monotone linear operator $\tilde{T} : C(S) \to C(S)$ such that

$$T(fK) = \tilde{T}(f)K, \quad \text{for all } f \in C_+(S) \text{ and } K \in \mathscr{K}.$$

Using the results of the previous section, we can prove the following result, which is a generalization of [6, Theorem 6].

Theorem 6.2. *Let $(\mathscr{C}, d)$ be a metric (or semi-metric) convex cone, and let $\mathscr{K}$ be a convex subcone of $\mathscr{C}$ such that, for some element $K_0 \in \mathscr{K}$, the following is true*: $d(\lambda K_0, \mu K_0) = |\lambda - \mu|$, *for all $\lambda \geqslant 0$ and $\mu \geqslant 0$. Let $\mathscr{F} \subset C(S; \mathbb{R}_+)$ be a classical Korovkin system. Then,*

$$\mathscr{G} = \{fK_0;\ f \in \mathscr{F}\}$$

is a regular Korovkin system in $C(S; \mathscr{K})$.

Proof. Let $\{T_n\}_{n \geqslant 1}$ be a uniformly equicontinuous sequence of $\mathbb{R}_+$-linear operators on $C(S; \mathscr{C})$ that are monotonically regular over $\mathscr{K}$. Assume that $T_n G \to G$ for all $G \in \mathscr{G}$.

Let $F \in C(S; \mathscr{K})$ and $\epsilon > 0$ be given. By the uniform equicontinuity of the sequence $\{T_n\}_{n \geqslant 1}$, there is some $\delta > 0$, which we may assume to verify $\delta < \frac{1}{3}\epsilon$, such that $d(F_1(t), F_2(t)) < \delta$, for all $t \in S$, implies $d((T_n F_1)(t), (T_n F_2)(t)) < \frac{1}{3}\epsilon$ for all $t \in S$. By Example 4.1, applied to the semi-metric convex cone $(\mathscr{K}, d)$, there exists an element in $C(S; \mathscr{K})$ of the form

$$t \mapsto \sum_{i=1}^{m} \phi_i(t) K_i,$$

where $\phi_1, \ldots, \phi_m \in C_+(S)$ and $K_1, \ldots, K_m \in \mathscr{K}$, such that

$$(1) \qquad d\left(F(t), \sum_{i=1}^{m} \phi_i(t) K_i\right) < \delta,$$

for all $t \in S$. Hence, for all $n \in \mathbb{N}$ one has

$$(2) \qquad d\left((T_n[F], t), \sum_{i=1}^{m} \left(\tilde{T}_n[\phi_i], t\right) K_i\right) < \tfrac{1}{3}\epsilon,$$

for all $t \in S$. Now $T_n G \to G$, for each $G \in \mathscr{G}$, implies that $\tilde{T}_n(f) \to f$, for each $f \in \mathscr{F}$. Indeed,

$$d(f(t)K_0, (T_n[fK_0], t)) = d\left(f(t)K_0, \left(\tilde{T}_n[f], t\right)K_0\right) = |f(t) - \left(\tilde{T}_n[f], t\right)|,$$

for all $t \in S$. Now $\{\tilde{T}_n\}_{n \geqslant 1}$ is a uniformly equicontinuous sequence of linear monotone operators in $C(S; \mathbb{R})$. Hence $\tilde{T}_n[g] \to g$ for all $g \in C_+(S)$. In particular, $\tilde{T}_n[\phi_i] \to \phi_i$ for each $i = 1, \ldots, m$. Hence, for some $n_0 \in \mathbb{N}$, if $n \geqslant n_0$, then

$$|\left(\tilde{T}_n[\phi_i], t\right) - \phi_i(t)| < \tfrac{1}{3}\epsilon(m(1 + d(K_i, 0)))^{-1},$$

for all $t \in S$. Therefore $n \geqslant n_0$ implies

$$d\left(\left(\tilde{T}_n[\phi_i], t\right)K_i, \phi_i(t)K_i\right) \leqslant |\left(\tilde{T}_n[\phi_i], t\right) - \phi_i(t)| \, d(K_i, 0) < \frac{\epsilon}{3m},$$

for all $t \in S$ and $i = 1, \ldots, m$. Hence,

$$(3) \qquad d\left(\sum_{i=1}^{m} \left(\tilde{T}_n[\phi_i], t\right)K_i, \sum_{i=1}^{m} \phi_i(t)K_i\right) \leqslant \sum_{i=1}^{m} d\left(\left(\tilde{T}_n[\phi_i], t\right)K_i, \phi_i(t)K_i\right) < \tfrac{1}{3}\epsilon,$$

for all $t \in S$. From (1)–(3) we get

$$d((T_n F, t), F(t)) \leqslant d\left(F(t), \sum_{i=1}^{m} \phi_i(t)K_i\right) + d\left(\sum_{i=1}^{m} \phi_i(t)K_i, \sum_{i=1}^{m} \left(\tilde{T}_n[\phi_i], t\right)K_i\right)$$
$$+ d\left(T_n\left(\sum_{i=1}^{m} \phi_i(t)K_i\right), (T_n[F], t)\right) < \epsilon,$$

for all $t \in S$, if $n \geqslant n_0$. Hence $T_n F \to F$ for all $F \in C(S; \mathscr{K})$. □

Corollary 6.3. *Let $(\mathscr{C}, d_H)$ be a Hausdorff convex cone, and let $\mathscr{F} \subset C(S; \mathbb{R}_+)$ be a classical Korovkin system in $C(S; \mathbb{R})$. Then,*

$$\mathscr{G} = \{fB;\ f \in \mathscr{F}\}$$

is a regular Korovkin system in $C(S; \mathscr{C})$.

Proof. In any Hausdorff convex cone $(\mathscr{C}, d_H)$ one has

$$d_H(\lambda B, \mu B) = |\lambda - \mu|,$$

for all $\lambda \geqslant 0$ and $\mu \geqslant 0$. It remains to make $\mathscr{K} = \mathscr{C}$ in Theorem 6.2. □

Corollary 6.4. *Let E be a normed space over the reals, and let $v \in E$ be chosen with $\| v \| = 1$. If $K_0 = \{v\}$, and $\mathscr{F} \subset C(S; \mathbb{R}_+)$ is a classical Korovkin system for $C(S; \mathbb{R})$, then*

$$\mathscr{G} = \{fK_0;\ f \in \mathscr{F}\}$$

is a regular Korovkin system for $C(S; \mathscr{K})$, where $\mathscr{K}$ is any convex subcone of $\mathscr{C}(E)$ that contains K_0. In particular, $\mathscr{G}$ is a regular Korovkin system for $C(S; \mathscr{K}(E))$ and for $C(S; \mathscr{C}(E))$.

Proof. We have only to verify that $d_H(\lambda K_0, \mu K_0) = |\lambda - \mu|$ and then apply Theorem 6.2. Now,

$$d_H(\lambda K_0, \mu K_0) = \| \lambda v - \mu v \| = |\lambda - \mu|. \qquad \square$$

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